THE TWENTY-EIGHTH REGULAR MEETING OF THE SAN FRANCISCO SECTION.

THE twenty-eighth regular meeting of the San Francisco Section of the Society was held at the University of California on Saturday, November 25, 1916. The total attendance was nineteen, including the following members of the Society:

Professor R. E. Allardice, Dr. B. A. Bernstein, Professor H. F. Blichfeldt, Dr. Thomas Buck, Professor G. C. Edwards, Professor L. M. Hoskins, Dr. Frank Irwin, Professor C. J. Keyser, Professor D. N. Lehmer, Professor W. A. Manning, Professor C. A. Noble, Professor E. W. Ponzer, Professor T. M. Putnam.

Professor D. N. Lehmer was elected chairman of the Section and presided at the meeting. Professor W. A. Manning was elected secretary and Professors Blichfeldt, McDonald, and Manning members of the programme committee of the Section for the ensuing year.

The following papers were presented at this meeting:

(1) Professor W. A. MANNING: "The degree and class of multiply transitive groups."

(2) Professor W. A. MANNING: "The primitive groups of class fifteen."

(3) Miss EDNA JOHNSTON: "Graphical determination of the nature of the roots of cubic and quartic equations."

(4) Miss MAURINE LABER: "Concerning the combinations of 36 elements 6 at a time."

(5) Professor A. F. CARPENTER: "Some fundamental relations in the projective differential geometry of ruled surfaces."

(6) Professor C. J. KEYSER: "A note concerning groups of dyadic relations."

(7) Professor L. M. HOSKINS: "The theory of strain of an elastic solid from a condition of great initial stress; with application to two cases of strain of a gravitating sphere."

(8) Mr. J. S. TAYLOR: "Complete existential theory of Bernstein's set of four postulates for Boolean algebras."

(9) Professor H. F. BLICHFELDT: "Linear groups which contain transformations having only two distinct characteristic roots."

The papers of Miss Johnston and Miss Laber were communi-

cated to the Society by Professor Griffin. Mr. Taylor was introduced by Dr. Bernstein. In the absence of the authors the papers of Miss Johnston, Miss Laber, and Professor Carpenter were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Manning presented the following theorem: Let n be the degree and μ the class of a multiply transitive substitution group G that contains a substitution of order 2 and degree μ; then if the transitivity of G is

double, $\mu > \frac{1}{2}n - \frac{1}{2}\sqrt{n} - 1$;
triple, $\mu \geqq \frac{1}{2}n$;
4-ple, $\mu \geqq \frac{1}{2}(n+1)$;
5-ple, $\mu \geqq \frac{3}{5}n$;
6-ple, $\mu \geqq \frac{3}{4}n$;
7-ple, $\mu \geqq \frac{10}{13}n$;

more than $(p_1 + p_2 + \cdots + p_r)$-ple, where $p_1, p_2, \cdots, p_r$ are distinct odd primes $(r > 1)$,

$$\mu > \frac{(p_1 - 1)(p_2 - 1) \cdots (p_r - 1) - 1}{(p_1 - 1)(p_2 - 1) \cdots (p_r - 1)} n.$$

Bochert, in the proof of his theorem, makes the commutator the corner stone of the whole structure. Here the commutator is abandoned and a close study of diedral rotation groups takes its place. The results of that study may be thus summarized: If the order of a group of class μ generated by two substitutions s and t of order 2 and degree μ is divisible by each of the odd primes $p_1, p_2, \cdots, p_r$, its degree n does not exceed $\mu + \mu/(p_1-1)(p_2-1)\cdots(p_r-1)$; if the order N of st is odd or is divisible by 4, n does not exceed $\mu + 2\mu/N$; if N (> 2) is twice an odd number, n is not greater than $\mu + 2\mu/N + 8\mu/N^2$.

2. It has been known for some time that the only primitive substitution groups of class $3p$, p an odd prime, that contain a substitution of degree $3p$ and order p are three in number and belong to one class only, class 15. In addition to these groups Professor Manning finds that there are just four other primitive groups of class 15. In fact, if k is any positive whole number greater than three, there are three simply transitive primitive groups of degree k^2 and class $3k$, and one simply

transitive primitive group of degree $\frac{1}{2}(k+1)(k+2)$ and class $3k$. The last is simply isomorphic to the alternating group of degree k. The others are simply isomorphic to the three imprimitive groups of degree $2k$ whose intransitive head is the direct product of two alternating groups of degree k in two different sets of letters, or is the dimidiate of the direct product of two symmetric groups of degree k on distinct letters. An auxiliary theorem suggested by this problem is this: If the degree of a transitive constituent of the subgroup leaving one letter fixed in a simply transitive primitive group exceeds by two (or more) units the degree of any other transitive constituent of that subgroup, then the constituent of highest degree is a simply transitive group.

3. In this note Miss Johnston points out a simple way to determine the nature of the roots of a quartic, in the various possible cases, by considering the points in the complex plane which represent the roots of the reducing cubic and their respective square roots. Similarly for a cubic and its reducing quadratic (sextic). Certain questions relating to discriminants are also discussed.

4. In his Mathematical Essays and Recreations, Ball describes methods of finding $n(n+1)$ combinations of n^2 elements taken n at a time without repetition of any pair, in the cases $n = 4, 5, 7, 8$. In the present note Miss Laber proves this to be impossible in the case $n = 6$, reported by Ball as having "baffled all attempts to find a solution." The proof is based upon a very convenient symmetrical system of notation, and a method of enumeration by which combinations involving common pairs can be rapidly eliminated. The number of admissible combinations resulting from various types of substitutions is noted.

5. In his projective theory of ruled surfaces, Wilczynski makes frequent use of a moving tetrahedron of reference, the homogeneous coordinates of whose vertices, when referred to any fixed system of homogeneous coordinates, are solutions of the system of differential equations

$$y'' + p_{11}y' + p_{12}z' + q_{11}y + q_{12}z = 0,$$
$$z'' + p_{21}y' + p_{22}z' + q_{21}y + q_{22}z = 0.$$

If a point, line or plane is to be fixed in space, its coordinates when referred to this moving tetrahedron must satisfy certain conditions expressible as differential equations, and conversely. Professor Carpenter's paper (*a*) determines necessary and sufficient conditions for fixed points, lines and planes and (*b*) indicates by a number of theorems the usefulness of these conditions in enriching the theory of ruled surfaces. A portion of this paper was communicated to the Chicago Section of the Society at the meeting of April, 1915.

6. The abstract relations studied in symbolic logic fall into various cardinal classes. The relations of a class are combinable in accordance with certain fundamental rules or modes, as logical addition, logical multiplication, relative multiplication. Professor Keyser's paper points out the desirability of examining the more important systems thus arising with a view to ascertaining which of them are groups. The paper presents some results found on the threshold of such an examination. It is found, for example, that under any of the mentioned rules of combination the ensemble of relations that constitute a type (as this term is understood in the Principia Mathematica of Whitehead and Russell) possesses the group property and two but only two of the other properties required in the four-property definition of a group. On the other hand, the ensemble of one-one relations that cover and are restricted to an arbitrarily chosen field of relations is a group if the rule of combination be relative multiplication, though the same ensemble does not possess even the group property if the rule of combination be logical addition or logical multiplication.

7. The earlier papers upon the strain of a gravitating elastic sphere were based upon a defective physical theory, in that the stress-strain relations were assumed to hold for a fixed element of space rather than for a definite element of matter. This defect was pointed out by Professor Hoskins in 1910 (*Transactions*, volume 11, page 504) and by A. E. H. Love in 1911 (Some Problems of Geodynamics, page 89). An amended theory was stated by Love and applied to certain problems including that of the tides in a gravitating sphere of uniform density. The present paper is based upon a theory in essential agreement with that of Love but formulated in a somewhat

more general manner. Professor Hoskins gives two applications: (1) to the problem of the strain of a gravitating compressible sphere of uniform density; and (2) to that of a gravitating incompressible sphere composed of a shell and nucleus of different densities. Both solutions follow the lines of the analysis employed in two former papers (*Transactions*, volume 11, page 203 and page 494). The same method can be extended to more complex cases such as that of a sphere composed of several shells and a nucleus, all of different densities and rigidities, such extensions being merely a matter of algebraic labor.

8. The complete existential theory of Dr. Bernstein's set of four postulates for Boolean algebras is shown by Mr. Taylor to consist of six propositions of non-existence and ten propositions of existence. The non-existences are expressed by the proposition $\Sigma^{-1} \supset \Sigma^{34}$. The propositions of existence are proved by the exhibition of ten systems Σ: two examples for K singular, six examples for K dual, and two examples for K triple. The postulates are shown to be completely independent in the sense defined by Professor E. H. Moore if the minimum number of elements of the class K be postulated as three.

9. Professor Blichfeldt proves that:

(*a*) No primitive linear group can contain a transformation of order $k > 5$ whose "multipliers" (characteristic roots) have only two values. For instance, no primitive group in three variables can contain the transformation $x_1 = x_1'$, $x_2 = \epsilon x_2'$, $x_3 = \epsilon x_3'$, whose multipliers 1, ϵ, ϵ ($\epsilon \neq 1$) are kth roots of unity; $k > 5$.

(*b*) If a primitive group G, say in n variables, contains a transformation S of order k, where $6 > k > 3$, having just two distinct multipliers, then these multipliers are each repeated $\frac{1}{2}n$ times. Moreover, G contains an invariant subgroup K. This is of monomial type and order 2^p when $k = 4$. If $k = 5$, then K is the direct product of say q groups of order 60φ and icosahedral type, and n must be divisible by 2^q.

W. A. MANNING,
Secretary of the Section.

THE TENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE tenth regular meeting of the Southwestern Section was held at the University of Kansas, at Lawrence, on Saturday, December 2, 1916. About thirty persons were in attendance, among them the following twenty-three members:

Professor C. H. Ashton, Professor A. A. Bennett, Professor Henry Blumberg, Professor W. C. Brenke, Professor P. J. Daniell, Professor E. W. Davis, Professor I. M. DeLong, Professor E. P. R. Duval, Professor W. H. Garrett, Professor W. A. Harshbarger, Professor E. R. Hedrick, Professor O. D. Kellogg, Professor S. Lefschetz, Dr. G. H. Light, Mr. W. A. Luby, Professor U. G. Mitchell, Professor Mary W. Newson, Professor S. W. Reaves, Dr. Paul R. Rider, Professor W. H. Roever, Mr. L. L. Steimley, Professor E. B. Stouffer, and Professor J. N. Van der Vries.

Attending members were very hospitably entertained at a smoker on the evening before the meeting, and at lunch between the sessions, at the University Club. It was decided that the next meeting of the Section should be held at the University of Oklahoma, at Norman, on December 1, 1917. The following programme committee was appointed: Professors S. W. Reaves (chairman), Henry Blumberg, O. D. Kellogg (secretary).

The following papers were presented:

(1) Professor A. A. BENNETT: "Newton's method in general analysis."

(2) Professor S. LEFSCHETZ: "Double integrals of the second kind belonging to an algebraic surface."

(3) Professor S. LEFSCHETZ: "On the residues of double integrals belonging to an algebraic surface."

(4) Dr. P. R. RIDER: "A note on discontinuous functions in the calculus of variations."

(5) Professor W. H. ROEVER: "Series for computing the roots of the equation $x + p \tan x = 0$."

(6) Professor E. B. STOUFFER: "On the calculation of invariants and covariants of linear homogeneous differential equations."

(7) Professor P. J. DANIELL: "The Lebesgue-Stieltjes integral."

(8) Professor HENRY BLUMBERG: "An example of a discontinuous function with a certain remarkable property."

(9) Dr. G. H. LIGHT: "Note on the relation between the focal points of an extremal when both end points are free."

(10) Professor G. C. EVANS: "An extension of Hadamard's formula for a linear functional."

(11) Professor H. C. GOSSARD: "On the relations between the faces and edges of a tetrahedron."

(12) Professor HENRY BLUMBERG: "Convex functions."

(13) Professor E. R. HEDRICK: "Functions that are nearly analytic."

Professor Blumberg's first paper and Professor Gossard's paper were read by title. Abstracts of the papers follow.

1. In this paper, Professor Bennett considers the formal character of Newton's method of approximating to a root of an equation, whether algebraic or transcendental. By generalizations of the notions sum, unknown quantity, coefficient, product, absolute value, etc., Newton's method is shown to apply also in a general field whose restrictions are considered explicitly. In particular, application is made to certain types of non-linear integral equations. The paper appeared in the *Proceedings* of the National Academy of Sciences, October, 1916.

2. The plan followed by E. Picard in his development of the theory of double integrals of the second kind is somewhat as follows: (*a*) reduction, largely by algebraic processes, of these integrals to a normal form; (*b*) development of the theory of integrals of total differentials of the third kind; (*c*) application of (*b*) to a special class of double integrals of the second kind; (*d*) investigation of the periods of integrals in the normal form; (*e*) enumeration of integrals of the second kind. In this note, Professor Lefschetz shows that this theory can be much simplified by starting with *d*. From this *a* and *b* then follow easily, and thus most of the algebraic work found in Picard's presentation is avoided.

3. A double integral belonging to an algebraic surface F may have cyclic or polar residues. It is shown in Professor

Lefschetz's second paper that the latter have the following two properties: (*a*) those with respect to points on the same algebraic curve have a zero sum; (*b*) those with respect to a given ordinary point of F also have a zero sum. The first follows at once from a well known theorem, while by means of quadratic transformations the second is reducible to a proposition given by Poincaré. Conversely, given a set of numbers corresponding in a suitable way to an assigned set of algebraic curves of F and satisfying the conditions *a* and *b* above, there is a double integral belonging to F of which they are the polar residues. No special conditions have to be satisfied by cyclic residues. This note will appear in the *Quarterly Journal of Pure and Applied Mathematics.*

4. In this paper, Dr. Rider gives the corner conditions and the Carathéodory Ω-function for discontinuous solutions in the calculus of variations for the form of the problem considered by Bliss in several papers, and for a new form of the space problem discussed by the author himself in a paper to be published in the *American Journal of Mathematics.*

5. For computing the roots x_n $(n = 0, 1, 2, \cdots)$ of the equation $x + p \tan x = 0$, Professor Roever gives two types of series. The first is a power series in p, the coefficients in the series for x_n being polynomials in the reciprocals of the number $t_n = (2n + 1)\pi/2$. The second type is a power series in $p + 1$, the coefficients being polynomials in the reciprocals of the numbers r_n, or the nth solution of the equation $x - \tan p = 0$. As $r_0 = 0$, a special formula is needed for the root x_0 in the second type of series.

6. Consider the system of linear homogeneous differential equations

$$(A) \qquad y_i'' + \sum_{k=1}^{n} (2p_{ik}y_k' + q_{ik}y_k) = 0 \quad (i = 1, 2, \cdots, n),$$

where p_{ik} and q_{ik} are functions of the independent variable x. In a paper presented to the Society at Chicago in April, 1915, Professor Stouffer calculated the complete system of semi-invariants of (A). In the present paper, he obtains the complete system of semi-covariants by the use of certain operators. He also sets up the systems of partial differential

equations which must be satisfied by any invariant or covariant of (A). These equations are expressed in terms of the seminvariants and semi-covariants. It is thus easy to test whether an expression in the seminvariants and semi-covariants which may arise in the geometry of the system is also an invariant or covariant.

7. The Lebesgue-Stieltjes integral is defined by Professor Daniell in this paper by means of sets of points in a way similar to that employed in defining the Lebesgue integral. It possesses the Lebesgue property that the limit of the integral is the integral of the limit. Like the Stieltjes integral, it is discontinuous, and the "measure" of even a single point may be finite. It is therefore suitable for use in physics, particularly electricity, where charge (or mass) may be used as a "measure" of a set of points. Most of the fundamental properties of Lebesgue integrals are shown to be satisfied by it. The paper will be offered to the *Transactions*.

8. Professor Blumberg shows with the aid of Zermelo's Wohlordnungssatz that there exist real, one-valued, discontinuous functions $f(x)$, defined in the linear continuum, such that if P is a given perfect set, then $f(x)$ is discontinuous with respect to P at every point of P.

9. In the *Mathematische Annalen*, volume 75, Rasmadse has considered the case of one fixed and one variable end point. He found that the focal point t_2'' comes before the focal point t_1'' if the integral $J = \int_{t_1}^{t_2} F(x, y, x', y')$ is a minimum. Dr. Light shows that t_2'' and t_1'' must coincide in the general case in which both end points are free.

10. In a recent paper in the BULLETIN (November, 1916) Dr. Fischer gives an interesting formula for a linear functional $T[\varphi]$ which has continuity merely of order k in its argument $\varphi(x)$, instead of order zero. Hadamard's form for a linear functional, which is itself slightly more general than the others, admits, according to Professor Evans, this extension in a very simple form. If T, depending on $\varphi(x)$ in the interval (a, b), has continuity merely of order k, it may be written: $T[\varphi(x)] = \lim_{n=\infty} \int_{a-\epsilon}^{b+\epsilon} \varphi(x)\Psi(x, n)dx$, where $\varphi(x)$ is extended

continuously with its first k derivatives beyond the interval (a, b) to the interval $(a - \epsilon, b + \epsilon)$, ϵ being a quantity arbitrarily small, but positive. For $\Psi(u, n)$ may be taken such a function as $\sqrt{n/\pi}T[\{1 - (u - x)^2\}^n]$. These results will probably be published in the Cambridge Colloquium Lectures.

11. By expressing the absolute in terms of the faces of a tetrahedron and also in terms of the edges, both equations being written in line coordinates, and then equating coefficients, Professor Gossard obtains (a) six equations of the type $(3V/2)^2 e_{12}^2 = A_{00}A_{33} - A_{03}^2$; ($b$) twelve of the type $(3V/2)^4 e_{12}^2 e_{20}^2 = (A_{03}A_{13} - A_{01}A_{33})^2 + 4A_{33}(3V/2)^4$; ($c$) three of the type $9V^2[(e_{01}^2 + e_{23}^2) - (e_{02}^2 + e_{13}^2)] = 16(A_{01}A_{23} - A_{02}A_{13})$, where V is the volume, A_{ij} the product of the areas of the faces opposite the vertices $e_i = 0$ and $e_j = 0$ by the cosine of the angle between the two faces, and where e_{ij} is the length of the edge from the vertex $e_i = 0$ to the vertex $e_j = 0$.

12. The real, one-valued function $f(x)$ of the real variable x is (according to Jensen) said to be "convex" in the interval (a, b), where it is defined if, for every pair of values x_1, x_2 of x, the inequality

$$f\left[\frac{x_1 + x_2}{2}\right] \leqq \frac{f(x_1) + f(x_2)}{2}$$

holds. Similarly, the function $f(x, y)$, defined in the square S, is said to be convex in S, if for every pair of points (x_1, y_1), (x_2, y_2) in S, the inequality

$$f\left[\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right] \leqq \frac{f(x_1, y_1) + f(x_2, y_2)}{2}$$

holds. Professor Blumberg shows that if the convex function $f(x)$ has at least one discontinuity between a and b, it is necessarily non-measurable in the Lebesgue sense. Furthermore, if the convex function $f(x, y)$, defined in the square S, is such that the set of functional values of $f(x, y)$ on the boundary of S has a finite upper bound, then $f(x, y)$ is continuous at every interior point of S. This result remains true if for "square" we substitute "closed Jordan curve," and, in fact, more general curves may be allowed. Generalizations to n dimensions are given.

13. In this paper Professor Hedrick describes a method of characterizing functions of a complex variable that are not analytic, but that differ only slightly from analytic functions. For example, the function $\psi(z) = f(z) + \epsilon\varphi(z)$, where $f(z)$ is analytic, and ϵ is a real number, will differ but little from $f(z)$, if ϵ is chosen small, at all points where $|\varphi(z)|$ is finite. The Riemann surface for such functions may be studied by means of its relation to that for $f(z)$. A series of examples of this type and of more general types is given.

O. D. KELLOGG,
Secretary of the Section.

THE RELATIONS OF MATHEMATICS TO THE NATURAL SCIENCES.

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 28, 1916.

BY PRESIDENT E. W. BROWN.

THE duty enjoined on the President of the American Mathematical Society of delivering an address at the close of his term of office is at once an opportunity and a danger. It is one of the rare occasions when he is able to discuss matters which are unsuited for a memoir and when it is proper to try to take a somewhat broader view of his subject than is suggested in investigations designed to elucidate some special part of it. In so doing, he necessarily must look into the future and attempt to foresee it through such indications of the present as may seem significant; and the danger of becoming a false prophet or of raising an unnecessary alarm is unattractive to anyone, least of all to those who have the lifelong habit of feeling their way into the unknown by roads slowly constructed and securely laid. I am willing to run this risk because I believe that there are certain matters connected with the future of mathematical science which need fuller consideration than they have received of late years. In discussing them, one must necessarily tread on debatable ground. It is, however, a happy custom to regard the matter contained in a presidential address not as an official presen-

tation of the views of the Society but solely as the personal opinion of the one member appointed to deliver it. Such suggestions as I have to make in connection with my topic "The Relations of Mathematics to the Natural Sciences" have no claim to novelty or originality. They are partly the natural result of association with those who work in more or less closely related fields and partly of a long connection with a task which has led in directions away from those usually followed by the modern mathematician. Before this discussion, however, a brief summary of the principal events of the past two years may not be out of place.

The summer meeting of the Society in Cambridge, as you are aware, was made the occasion for a celebration of our twenty-fifth anniversary. Originally started in 1888 as a meeting of those interested in mathematics in New York and its vicinity, the Society emerged three years later in the form of a national organization with a BULLETIN containing a record of the activities of its members and with some limited publication of their work and ideas. Nine years later, the growth of research in America, unquestionably assisted by the opportunities afforded by the Society, caused a need of further facilities for publication and the *Transactions* was started. Both journals have grown beyond the sizes originally contemplated and we may hope or fear that a new medium for publication may be required in the not distant future. The original meeting place in New York City has been quite insufficient to supply our members with opportunities to present papers and to engage in discussions; one after another, sections have been founded in various parts of the country which hold sessions at regular intervals and which assist actively in contributing to the growth of the subject and the welfare of the society. Our library numbers over 5,000 volumes, besides a large collection of unbound dissertations. And finally the offices of the Society and most of its records have survived the fire which seems to be a necessary preliminary to permanent stability.

This growth has lately caused a serious problem to arise in connection with the extent of the activities of the Society. Many of those who on account of professional duties or for other causes are unable to follow or lead far along the high-roads of research, have felt that there are many byways in mathematics which are of interest and which would stimulate

thought. They felt too that some centralization of efforts to improve the teaching of mathematics was necessary. The Society had hitherto confined its main activities to research and it became a grave question whether it should undertake new duties and responsibilities in addition to the already heavy burdens laid on certain of its officers. The difficulty has been happily solved by the formation of the Mathematical Association of America with assumption of responsibility for the *American Mathematical Monthly* and its use as the official journal of the Association. This result has been brought about by the cordial cooperation of everyone concerned. They have desired not only that the interests of the American Mathematical Society should not suffer but that the new Association should be so organized and conducted as to assist in strengthening and coordinating all forms of mathematical activity throughout the country. Our best wishes go out to our young and already vigorous offspring.

In outside activities, the Society during the last two years has appointed representatives to the general Committee on the classification of technical literature, to the Committee on standards for graphic representation and to the Naval Advisory Board. In spite of some misconception in the minds of those who were responsible for the invitation to the last named, as to the functions of mathematicians, both the Society and the Board are to be congratulated on the work of our two representatives.

Perhaps the most important outside movement affecting the chief objects of the Society is that initiated by the National Academy of Sciences for the organization and development of research. Originally started with the object of assisting in the defence of the country, it enlarged its functions when it was seen that much would be gained through cooperation and mutual assistance not only by investigators themselves but by all those who need and employ investigators in technical and manufacturing processes. This is therefore a movement in the interests of the whole community. The Society has already, through the Council, expressed its desire to cooperate. We may hope that there will result some benefits for our science and that we shall be able to furnish in turn some assistance toward the solution of the problems presented by other sciences.

With this record of achievement in the past it is but natural

that we should look forward to the future and note whether, besides promise of progress and success, there are indications of danger against which measures of protection or alleviation are necessary.

Mathematics may be treated as a science or art which holds within itself all that is necessary for its own development. It may also be treated as an aid for the development of the natural sciences and for all forms of investigation which admit of deduction from exact statements. In the one, an advance in knowledge of the subject itself by examination of the interrelations of a set of axioms is sought; in the other, it is used to deduce phenomena of nature as the consequences of physical laws which have themselves been deduced by observation or experiment. In the earlier history of the subject, the difference between the two points of view was but little recognized. The axioms themselves were mainly suggested by natural phenomena and the developments were principally those needed in the elucidation of the phenomena. As time went on, mathematics was allowed to progress in its own way, unhindered by any necessity for present or future applications. This freedom from a somewhat entangling alliance has resulted in a marvellous structure of thought. The fundamental bases of the subject have been clarified and organized, the modes of reasoning have become subject to careful scrutiny and every effort has been directed to make sure that the structure built on these lines shall be without blemish.

There is, however, a reverse side to the picture. While every other product of human thought demands aid from outside, modern pure mathematics stands practically alone. Its laws, its logic, what it seeks and what it finds have no necessary relation to externals in the sense of dependence. And this independence has been achieved by no ordinary labor and by the exercise of many minds possessing gifts of insight in the highest degree. Yet one cannot help asking whether it is for the best interests of the subject that it should continue in this isolation. An external stimulus seems to be necessary, at least from time to time, to produce the best elements of growth in most forms of human activity. Is the continued development of pure mathematics an exception to the rule? One may argue that isolation produces a pure strain, but is it not also true that an occasional crossing of the breed is necessary to prevent the species from running itself

out? I do not wish to imply that this is a real analogy; whether it is so or not, it can at least serve as an illustration of the manner in which the separation may affect mathematics. If it be admitted that future injury is likely to result, it is obvious that we should search for methods to alter the condition; if the contrary is true, I can still fall back upon the demand that other sciences make for assistance.

At the outset one must admit that the separation is not altogether confined to the relations of mathematics to the natural sciences, it has been growing even between its different branches, not so much perhaps in essentials as in the fact that there are only a few amongst us who can follow except in a dim way or with much labor, the larger part of the advances which are being published so frequently. This is partly due to the considerable body of knowledge which has accumulated: time and opportunity are lacking to assimilate even a small portion of it. And there are other contributing causes which will occur to you. Over one of them, of minor importance perhaps but in which some improvement may be possible, I shall delay for a moment.

The symbolism employed to convey ideas in mathematics is unquestionably good in its general outlines and serves the purposes of the study well. It is brief, it admits of considerable variety, and it is easily adapted to most of the needs of the mathematician. Unfortunately, few standards have been kept. While the general principles by which the notation is to be used are little changed, the widest variations in the meanings of different sets of symbols have been allowed to creep in; and conversely, the same sets of ideas have been frequently represented by a variety of symbols. In my own subject of celestial mechanics, this variety continually halts one's thoughts although the mathematical ideas are comparatively simple; in fact the symbols rarely represent anything else than the most elementary operators and quantities which are either fixed or have continuous variation. But the difficulty is far greater where the ideas themselves are complex, where a single symbol may imply numerous properties and where different authors adopt different meanings to be attached to the same symbol. The perfectly simple triad of marks " (1)," for example, may represent a number, an operator, a group, a function, an axiom, a convention—and any one of these may have special properties attached or particular limitations

imposed. And when to this are added many other symbols with similar extensions of meaning, one difficulty of mastering a memoir of average length becomes sufficiently obvious. The language must first be learned, and a new language in general confronts the reader when he turns to some memoir by a different author, even on the same subject.

Is there a remedy for this condition? Can mathematical notation be standardized to some extent and if so, can it be done in such a manner that it will meet with general acceptance and use? It is perhaps a Herculean task but I am not at all sure that a concerted effort would not result in some improvement.

With what material have we to deal? On the one side there are the four double fonts of the Roman, Italic, Greek, and German alphabets, one or two fonts of the Arabic numbers and a few well-known symbols, the types for which exist in any good printer's composing room. On the other side there is a considerable body of ideas to be represented, although the number of those most frequently used is not very great. Suppose, by way of illustration, that some very general conventions were agreed upon: for example, that operators should be denoted by capital letters, except in the few cases like those of differentiation or integration, where the notation is already standardized; continuously varying quantities by certain groups of lower case letters, discontinuous quantities like those in our number system by other groups, quantities remaining fixed by still other groups. Continuing with this idea we should represent those properties or limitations which are of frequent occurrence by certain combinations of marks and letters and it would be advisable to agree that the older signs, those of the four chief arithmetical operations for instance, shall retain their original meanings if unattended by any distinguishing mark. Any such plan should depart as little as possible from the schemes of notation which have been most generally used. One result undoubtedly would be some loss of the brevity which now characterizes the mathematical symbolism of much of the published work. I doubt, however, whether this would be an evil, in fact, it might be an advantage; the reader would soon become familiar with many groups of symbols and would read their meaning without effort, just as he recognizes a printed word not by the separate letters but by the appearance of the group of letters which form the

word. After all, brevity is not the soul of mathematics. If a scheme with such broad outlines is not feasible, it might still be possible to apply the method separately to divisions of mathematics. But this idea is less attractive and presents additional difficulties in the overlapping of different divisions. Most of us would prefer, I think, to learn one language with many words, than several different languages even though their vocabularies were much smaller.

It may be objected that any standardization of symbolism may seriously interfere with the freedom of the investigator and so limit his output. One may ask in reply whether past liberty has not already degenerated into license and whether some regulation will not confer benefits on mathematics similar to those we acknowledge in the framing of laws for the welfare of the community. If the circulation of the output be increased we can well afford some decrease in the quantity.

It is a much more simple matter to suggest a method for bringing any such scheme or schemes into operation if once agreed on by an authoritative committee. After proper and frequent publication, there is no need for any kind of compulsion; a writer, using a settled scheme has merely to say so. In this way it would either come into general use or die a natural death without having caused any particular harm. There is another consideration in this connection. We must look forward to the time when our turn will come to publish a revised encyclopædia of mathematical sciences; if standards of notation are then available, one can hardly overemphasize their value in making a summary of the work of the past.

In the meantime some assistance can be given by such aids as tables of notation and separate summaries attached to each paper published, even at the cost of artistic appearance; this has already become a feature of some journals in other subjects. And there are probably other devices which might usefully be employed to assist the reader.

These are, however, mechanical aids and only touch on the fringe of the question which is my principal theme—the isolation of mathematics from the natural sciences. There is, I must confess, at present little prospect of the cure generally proposed, namely that mathematicians turn their attention to the applications. And there is good reason why this is not acceptable, at least in the form usually stated. To make progress in any one direction requires a more considerable

knowledge of the physical laws and past achievements in that direction than most of those who are making advances in their own lines have time to acquire. They may do much by encouraging a certain proportion of the students who come under their influence to attack such problems. The welcome given by the mathematical journals in America to memoirs on the applications is a sufficient answer to those who would believe that the study of the latter is discouraged. In this connection it is noticeable that the pure science has a tendency to drive out the applied science—this has been peculiarly evident in England during the last quarter of a century—or to put the matter in another form, the cleavage between pure mathematics and the experimental sciences which use mathematics has tended to increase.

It is necessary here to distinguish between the study of the mathematical methods used in the applications and the study of the applications by mathematical methods. In the former, the foundation, the logic and the limitations which must be imposed on the processes adopted by applied mathematicians are considered; in the latter, it is new phenomena or the coordination of known phenomena which are sought. Poincaré, who produced much in both directions, obviously kept the distinction clear in his own mind, even when he mingled the two in a single memoir like his treatise on celestial mechanics. And his point of view in the former was not that of criticism but of justification of the methods used in the latter. Unfortunately, while a study of methods clarifies and expounds, it is not as a rule fertile in producing new advances. It is only when the mathematician leaves aside the particular applications of a method and studies the latter in all its bearings that he is able to obtain novelty.

The study of physical laws and phenomena by the methods of mathematics stands on an altogether different basis. Why is it that with calls in every direction for help so little is done? One hears blame cast on the mathematician for this state of affairs and perhaps to some extent there is justification for it. But I believe that the blame must be shared at least equally by the physicist or engineer who asks for the solution of his problems. The latter are generally presented in such a manner that the mathematician who is not thoroughly familiar with the subject cannot even make a beginning; often he is not given the necessary information about the physical cir-

cumstances which will enable him to formulate the problem. In its generality, it may be quite beyond the present state of mathematics to give a solution and yet if he is allowed to neglect certain factors and to modify others it may be possible to obtain the main facts needed. But he cannot be expected to know all these details. The more difficult the problem, the more necessary is a full and complete statement of everything connected with it. The physical laws which may be assumed and which cover the principal features, the modifying circumstances which may be left aside if the problem is too difficult, the range of numerical values which constitute the data, the range of numerical values which are needed in the solution—all these should be carefully set forth in any statement of the problem.

It is this work of formulation that is mainly contained in the treatises on applied mathematics so far as the older and more simple problems suggested by natural phenomena are concerned. They also give the principles which should be followed in all cases. Thus the knowledge of the applied mathematician, using the word in the old sense, is necessary as the intermediary between the experimentalist and the pure mathematician if one is to help the other. He must have a considerable knowledge of the phenomena and with it a sufficient knowledge of mathematics; neither can be neglected. It is the almost complete disappearance of this type of scholar from among us that is perhaps responsible more than anything else for the isolation of which I have spoken. The temptation either to proceed to experiment or to turn to purely mathematical researches has been too strong.

The need of a remedy is obvious and in searching for it we may perhaps have to consider a change in our system of regarding the doctor's degree with its original thesis as a necessary preliminary to every opening which leads to scientific work. It is frequently stated that the main difficulty in obtaining applied mathematicians is the difficulty of finding a suitable thesis for a student, by an instructor who has not himself investigated in those lines, and we have few here who have done so. This is unquestionably true. Under the circumstances, those organizations which need such men for their own purposes, whether they be government bureaus or commercial houses, must create positions for applied mathematicians with the same advantages that they give to those who

perform experimental researches. They must understand that the type of man needed must be trained for the purpose, that he cannot usually be solely a physicist or solely a mathematician but must have received a sufficient education in both directions. If the demand is clearly set forth, there is little doubt that the supply will be forthcoming.

The necessary link once supplied, the pure mathematician can render valuable services. But to do so, he will need to change to some extent the methods which he is accustomed to employ. The generality and completeness which he is accustomed to give to his researches are rarely useful in the applications. What is needed is the discovery of solutions which in a limited range can be reduced to numerical values without an excessive amount of labor. Singularities, for example, are usually excluded for one cause or another; places where the function is regular are those commonly needed. If the constants expressed by symbols are so numerous that the problem is intractable, some of them must be given numerical values; it will frequently happen that all needed information can be obtained from two or three well chosen numerical cases. And it is not only the material stored up in the past that will be used; new devices will continually be needed and I do not think that he will find the problems any less worthy of his best efforts than those to which he has been in the habit of directing his attention. The chief difference will perhaps consist in a limitation to work in definite directions instead of roaming in the field of discovery with freedom to investigate wherever the prospect seems to attract.

It follows that some organization of the different processes needed in the applications of mathematics has become necessary. The simple problems and a very few of the difficult ones have been solved and progress is halted in the face of a demand which is greater than ever before. Whether one considers questions of purely scientific interest like those which are presented in celestial mechanics or questions of practical importance, it is still true that there are few workers and even with their combined efforts they cannot achieve much. If we are to enlist the services of others, it is necessary that the work be divided in such a manner that contributions to its various parts can be made by those who have the proper training and qualifications. The first question suggested by what has been said is to ask what part can be taken by those who have a training in pure mathematics.

It is always safe but not always useful to restrict oneself to generalities. I am fully aware of the danger which attaches to the mention of specific problems; they have been solved, or they are insoluble with known methods, or they are not suitable for mathematical treatment. Nevertheless, I am going to mention briefly two or three which I believe cannot properly be placed in any one of these excluded classes. So far as I know, there is no published systematic attempt to treat them in such a manner as to furnish results in a form available for actual application. Too much emphasis can scarcely be laid on the last point if the researches of mathematicians are to be connected with the applications. Much of such work may be tedious and uninteresting but there is also much which demands the exercise of the best powers which an investigator may possess. He is also in the most favorable position to do it. He has usually looked at his subject from many directions before setting forth that one which is best suited for a theoretical exposition and he will know and be able to develop most easily that which is suitable for the applications.

Of the fundamental requisites which need systematic examination, perhaps that of most practical value is an inquiry into the symbolic forms and modes of expression which are best adapted for numerical calculation. On the one side there exists a wealth of material in the shape of formulas, theorems, and methods available for choice. On the other side the machinery to which these are to be adapted is limited. Practically, the only numerical operations which can be performed mentally are addition, with its counterpart subtraction, and those operations in which a knowledge of order of succession in our decimal number system can be of assistance. All other operations are performed by means of tables whether memorized like the multiplication table or printed. Mechanical and graphical devices to replace these are available but the only one which has complete accuracy is that which replaces the mental operations, namely, the adding machine. The number of printed tables at our disposal is so small that a classification of these containing an account of their contents and of the uses to which they may be put is not a serious undertaking as a first step. Finally, the various devices by which the labor of numerical computation may be diminished should be collected; this is more troublesome because many of them are the results of experience and either have not been

published or are only mentioned incidentally in connection with special investigations.

It is a much less simple matter to try to point out a method of procedure for the adaptation of symbolic forms so that full use may be made of this numerical machinery. Something might certainly be achieved by gathering together those forms in most common use in a given subject or group of subjects and making a study of them from this point of view. An example occurs in the case of Fourier series; a number of papers have been published during the present century on the methods for connecting a Fourier series with its numerical values although most of the attention given has been in the direction of analysis rather than synthesis except in the case of purely mechanical devices. One may perhaps look forward to the construction of a more general theory which shall have as its object the formulation of laws or rules which will serve in all cases. Adaptation to numerical computation and the performing of the computations themselves are by no means the simple matters which they are frequently assumed to be if they are to be efficiently done. And it is this efficiency which must be the goal.

The converse problem of reducing a series of numerical values which obviously obey some law to symbolic form is equally in need of systematic treatment. If the law is known or there are reasons for the assumption of a specified form, the method of least squares enables the unknown constants to be determined. But it is rare that one opens a series of memoirs containing investigations in the physical sciences without seeing diagrams for which no law is known; in many cases no attempt is made to find one. The author has perhaps drawn a smooth curve which he judges would most nearly represent the facts if the observations had been freed from the errors which are incidental to all experimental work. But progress is halted until the curve can be so expressed that the relation of the physical phenomena that it represents to other phenomena or to known physical laws can be obtained. The best method of doing this is usually through a symbolic formula. The mere fact that functions with limited variation (and the known phenomena possess this property) can be represented by a Fourier series is usually of little or no assistance. Even if it be so represented it conveys almost no further information unless the phenomena themselves are of a periodic nature.

The Fourier series might perhaps be used as a first step in the investigation but there arises the difficult and practically unsolved problem of finding the function represented by the Fourier series, as was pointed out in a recent address. Some more direct and practical method is needed. In certain cases a mode of procedure can be indicated. When the curve appears to resemble the graph of some known function, it should be possible to devise methods of approximation by which additions to or changes in the function can be obtained, the constants present being then determined by the method of least squares. Transformations of functions so obtained involve the study of forms numerically equivalent within a given range to a given degree of accuracy.

The case of periodic or quasi-periodic functions deserves special mention. Light-curves of many stars are now being obtained with considerable accuracy. Some of the curves differ from a simple sine-curve so greatly that the representation by Fourier series will probably involve the presence of a large number of terms with sensible coefficients. Such forms will probably require transformation into other periodic functions if any clue is to be obtained to the laws which govern these changes. Our own sun is no exception. We obviously have to deal with forces which do not produce regular periodicity, but until some analytic forms can be obtained which will represent the curves with sufficient brevity and accuracy there is little hope of unravelling their mysteries.

I have mentioned these problems particularly because they lack the definiteness in the results demanded which characterizes most of the usual problems of mathematics. I do not think they belong to the subject less for that reason. Mathematics certainly can lay claim among its other properties to be a method of organizing ideas, and it has effectively done so in the past. The result of tossing a coin into the air is, owing to our lack of ability to work out all the consequences of the laws of mechanics, quite as indefinite, and yet mathematics has built up the solid and definite set of rules which we know as the theory of probabilities—a theory which is in practical use in every insurance company and in all scientific work: it is difficult to imagine what modern astronomy would be today without the method of least squares. Such problems as I have suggested may possibly have only temporary value but if in their solution some order is produced from chaos, a

contribution of practical use, at least in our time, will have been made.

Perhaps the set of mathematical problems most important for the applied sciences are those included in obtaining numerical solutions of differential equations. In many cases the ordinary methods such as finding exact solutions in terms of known functions or by means of series fail. The failure in the latter case is not a mathematical failure, but a practical one, owing to the large amount of numerical computation needed. Where it has to be undertaken, the particular method of analytical continuation of series, known as mechanical quadratures, is adopted. This certainly achieves the object in view but usually with a lack of real efficiency which cannot fail to strike anyone who attempts to use it to any extent. And it naturally becomes the more troublesome whenever the curve has any special feature such as near approach to a singularity. Questions which seem unimportant in theory sometimes loom large in calculation. For example, how many places of decimals at the start and throughout the work are needed to secure a given standard of accuracy? Is the same degree of accuracy necessary throughout? How large may each arc be taken? And so on.

If one examines textbooks or treatises for information, it is difficult to find anything of real use; in fact, the method as applied to the solution of, for example, a pair of ordinary simultaneous differential equations, I have not seen mentioned. In the form given in the investigations where it has been used, it is doubtless capable of great improvement. We may further ask: under what forms can it be used so that each arc shall be as long as possible with the same degree of accuracy? Is it necessary to confine it to power series and can it not be developed for computation in series of other functions so that, in any special case, one shall have a choice which can be made according to circumstances laid down in advance? Unquestionably there is material at hand to answer some of these questions if it could be worked up into a form simple enough for a person not possessing an intricate knowledge of differential equations and the properties of various modes of expansions to use. I hesitate in mentioning solutions of partial differential equations in this connection on account of the many inherent difficulties. Nevertheless the new problems arising in aviation and the older set connected with the motions

of our atmosphere and the prediction of the weather perhaps transcend in importance all others.

These problems are fundamental in the applications of mathematics. The expert knowledge and insight of the student of pure mathematics would seem to point him out as the one most likely to achieve at least partial successs in dealing with them. At the same time, little or no knowledge of physical processes and results is required. But if he happens to possess special information in any one of the natural sciences, I have little doubt that he can find a field giving a stimulus for new work and an opportunity to connect his subject with the outside world. Even biology is now demanding help in certain directions, somewhat reluctantly it is true, but none the less surely. Even if mathematics cannot always assist in solving problems, it can still help through its advantages as a brief and clear mode of description where processes are complicated and interdependent. But whatever is done in any direction, the work will be of little practical value unless it satisfies the final test, namely, that the form in which it is left is such that it may be applied without a full knowledge of the processes by which it is reached.

It is perhaps easy for one who has spent half his life in an undertaking which must always have numerical results as the final test, to exaggerate the importance of all work of this nature. This particular problem, it is true, has a variety which is not common to many such undertakings. Perhaps the most unexpected result has been that nearly all the portions which can be said to possess some small degree of novelty have arisen from the necessity of obtaining formulas which should give numerical results more easily than the older developments. And it has not infrequently happened that the actual computation of a particular case has indicated immediately the method which should be followed in the general case. This fact may not be a recommendation to the person who has the desire to use his powers to their fullest extent, yet for one who has a definite end in view and who wishes to reach the goal by the shortest route it may appeal as having certain advantages. Moreover, the time spent in actual computation is usually not a large fraction of that necessary for the whole work; if there are many computations of the same nature, each one may take on the average only a fifth or a tenth of the

time that an isolated case would require. The construction of numerical tables is a striking example of this as soon as the work has been properly organized.

Much of the mathematics used in the applications is repulsive to the pure mathematician, and properly so. It lacks the form and symmetry which so often assist in showing the ideas which are in process of development. There is a good reason for this. For efficient numerical application, many properties possessed by the initial formula and particularly that of symmetry should be used to assist; in the process of transformation they must necessarily disappear. The limitations, too, which are often of greater interest, are usually so much broader in the mathematics than in the applications demanded that there is no need to examine them. Or if they are reached, the numerical work at once warns of their approach and the formula or method is changed. Unfortunately, too, the very devices which constitute a good formula or by which a solution is reached often admit of such simple mathematical description that the air of mystery disappears and there is little external evidence of any special ingenuity. Nevertheless there is a degree of mental satisfaction in the achievement of concrete results which is a sufficient reward to those who work for them.

In this attempt to emphasize the need for more attention to the applications of mathematics, it is hardly necessary for me to deny a possible implication that the further study of the pure science is unprofitable. On the contrary, there seems to be more need for it than ever. But I believe that this need demands studies which shall take directions more immediately helpful to other sciences, not only for their assistance but for the sake of the subject itself. We cannot leave such matters to the chances of the future; some definite effort is required to work towards a goal which has been neglected in the past. It demands, it is true, limitations on one's freedom to follow attractive openings, it may require tedious and uninteresting developments, but these will not be deterrents to anyone who has previously attempted to advance knowledge without hope of external reward. Moreover, the natural bent of a nation which has been compelled by its environment to regard most of its affairs from a thoroughly practical point of view, gives promise of success. An incentive and the training once supplied, there seems to be no reason why a school of applied

mathematics of the first rank should not flourish on this continent.

Finally, any discussion which has investigation as its theme is incomplete if it leaves out of consideration the events which are taking place among the older nations. Although as a body and as individuals we have perhaps been less affected than others engaged in intellectual pursuits, nevertheless, these events cannot fail now and more so in the future to modify —perhaps to change entirely—our outlook. While the need for scientific research has been emphasized and brought forward more strongly than ever before, and in this movement we hope to share, yet it is to be remembered that a practical end has been visible throughout. Little stress has been laid on its purely intellectual side and we, in common with others engaged in research in pure science, may fear neglect and increasing lack of support. For a time at least, the problems of reconstruction and adaptation to new conditions will demand of the older nations most of the time and energy which has in the past been allowed to run in less practical routes. It is true that here we are happily removed from the devastation which has overtaken Europe, but the lack of stimulus from abroad cannot fail to have its effect. We need at this time to take stock of the resources of ourselves and of our Society, both mental and physical, to find whether they are sufficient for what may come and more than all to conserve the heritage which we have received from the old world so that it may later be returned alive and productive.

The possibility of great changes coming rapidly and with little warning must be faced. We cannot therefore be content to sit back and await events. The future of research in pure science is in danger as never before and the less practical the science the greater the danger. New economic and social forces are at work in every direction and may call many of us to activities which may stop temporarily and perhaps permanently the work and thought which have in the past most fully occupied us. Two generations of steady and continuous development have tended to make us settle in grooves of thought and habits of mind in which great upheavals have no place. To such changes at least a proper mental attitude is necessary. Action may not be needed now but preparation for action is essential and the chances of the future should be in the mind of every individual if the right course is to be pursued

when the time for action comes. Even at the present moment we can no longer regard the Society merely as one to forward mathematical research in America; the wider interests of the subject have now to be considered, since we form the one large national organization devoted to research in mathematics which can continue its work unhindered by more pressing claims. In these wider interests our opportunities for service may come: the readiness and ability that we show in a realization and fulfilment of them will be the chief measure of our success in the past and promise for the future.

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A THEOREM ON THE CURVES DESCRIBED BY A SPHERICAL PENDULUM.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society, December 27, 1916.)

1. IN what follows I shall restrict myself to the case considered by Greenhill,* in which the pendulum bob reaches (but does not go above) the horizontal plane of suspension with a non-vanishing horizontal velocity.†

When this assumption is made, there exists an interesting relation between the curve described by the bob and a certain quartic in space. It is the purpose of this note to establish this simple relation.

2. Let r_0 be the distance of the bob from the vertical axis when its horizontal projection crosses the x-axis; $A + 2\pi$ the horizontal angle between two consecutive positions in the plane of suspension; $4K$ the real period in the Jacobian elliptic functions appearing in the solution; $v = t/2w_1$, where w_1 is the real half-period in the Weierstrassian form, and t the time; a, b, c $(a > b > c)$ the real roots of the cubic in the general solution; $\mu = \sqrt{(l^2 + a^2)/(l^2 - a^2)}$, where l is the radius of

* Greenhill, Les Fonctions elliptiques et leurs Applications, chap. III. Appell, Traité de Mécanique rationnelle, t. 1, p. 494. Tannery et Molk, Eléments de la Théorie des Fonctions elliptiques, t. 4, p. 188.

† The statement by L. Lévy, on page 161 of his Fonctions elliptiques, that in this case x, y, z are doubly periodic functions of the time is not correct.

the sphere. As in this case $b = 0$, we find, according to Tannery, loc. cit., the positive sense of the z-axis being downward:

$$(1)\quad x = r_0[\cos(A+\pi)v \cdot \mathrm{cn}(2Kv) - \mu \sin(A+\pi)v \cdot \mathrm{sn}(2Kv)\mathrm{dn}(2Kv)],$$

$$(2)\quad y = r_0[\sin(A+\pi)v \cdot \mathrm{cn}(2Kv) + \mu \cos(A+\pi)v \cdot \mathrm{sn}(2Kv)\mathrm{dn}(2Kv)],$$

$$(3)\quad z = a\mathrm{cn}^2(2Kv),$$

for the coordinates of the bob in parametric form.

From (1) and (2) it is easily seen that

$$(4)\quad x\cos\phi + y\sin\phi = r_0\,\mathrm{cn}(2Kv),$$

$$(5)\quad -x\sin\phi + y\cos\phi = \mu r_0 \mathrm{sn}(2Kv)\mathrm{dn}(2Kv),$$

where $\phi = (A+\pi)v = (A+\pi/2w_1)t$ is an angle proportional to the time.

If we now put

$$(6)\quad x' = r_0\mathrm{cn}(2Kv),$$

$$(7)\quad y' = \mu r_0 \mathrm{sn}(2Kv)\mathrm{dn}(2Kv),$$

$$(8)\quad z' = a\mathrm{cn}^2(2Kv),$$

it is seen that the locus of the points (x', y', z') is a quartic in space. For each of the functions $\mathrm{sn}(2Kv)$, $\mathrm{cn}(2Kv)$, $\mathrm{dn}(2Kv)$, has $iK'/2K$ and $(iK'+2K)/2K$ as the only two poles within the period parallelogram, so that any linear combination

$$\alpha x' + \beta y' + \gamma z' + \delta$$

is an elliptic function with four poles (counting multiplicities properly), and consequently four zeros within the period parallelogram. The curve is therefore cut in four points by every plane and is a quartic.

The coordinates of a point $P(x, y, z)$ of the pendulum curve and the corresponding point $P'(x', y', z')$ of the quartic are connected by the formulas

$$(9)\quad \begin{aligned} x' &= x\cos\phi + y\sin\phi, \\ y' &= -x\sin\phi + y\cos\phi, \\ z' &= z. \end{aligned}$$

Hence, as x, y, z are functions of the time, the point P' is obtained by rotating the point P (in the negative sense) about the vertical axis (z-axis), through an angle which is proportional to the time t, with $-[(A+\pi)/2w_1]$ as a proportionality factor. Hence the

THEOREM: *If the points P of the curve of a spherical pendulum motion, which just reaches the horizontal plane of suspension with a non-vanishing horizontal velocity component, are rotated in the negative sense about the vertical axis through angles proportional to the times (with a definite proportionality factor) associated with those points, the locus of the rotated positions of the P's is a quartic in space of the first kind.*

4. The elimination of v between (6) and (8) leads to the equation of the parabolic cylinder

$$ax'^2 - r_0^2 z' = 0. \tag{10}$$

It touches the plane of suspension along the y-axis, it is below this plane, and is symmetrical with respect to the yz-plane. The elimination of v between (7) and (8) gives the equation

$$a^2y'^2 + \mu^2 r_0^2 k^2 (a-z)^2 - \mu^2 r_0^2 a(a-z) = 0, \tag{11}$$

which represents an elliptical cylinder. It cuts the z-axis at a distance a below and at a distance $a[(k^2-1)/k^2]$ above the origin. The quartic is now obtained as the intersection of these two cylinders and consists of only one real branch. The horizontal projection of this curve is a plane quartic of deficiency 1, and with the equation

$$\mu^2 k^2 x'^4 + \mu^2 r_0^2 (1-2k^2) x'^2 + r_0 y'^2 - \mu^2 r_0^4 (1-k^2) = 0. \tag{12}$$

The infinite point of the y-axis is an isolated point which must be counted for two double points.

The converse of the theorem is evidently also true, i. e.,

There always exists a quartic of the first kind in space, such that the rotations of its points about the vertical axis through certain definite angles proportional to the time, will lead to a given spherical pendulum curve, as specified above.

UNIVERSITY OF ILLINOIS.

A THEOREM CONCERNING CONTINUOUS CURVES.

BY PROFESSOR ROBERT L. MOORE.

(Read before the American Mathematical Society, October 28, 1916.)

In this paper I propose to show that every continuous curve has the simple property stated below in Theorem 1. Though my proof is worded for the case of a plane curve, it is clear that with a slight change in phraseology it would apply to a curve in a space of any number of dimensions.

LEMMA. *If S_1, S_2, S_3, $\cdots$ is a countable sequence of connected,* bounded point sets such that, for every n, S_n contains S'_{n+1},† then the set of all points that are common to S_1, S_2, S_3, $\cdots$ is closed and connected.*

For a proof of this lemma see my paper "On the foundations of plane analysis situs," *Transactions of the American Mathematical Society*, volume 17 (1916), page 137. Cf. also S. Janiszewski and E. Mazurkiewicz, *Comptes Rendus*, volume 151 (1910), pages 199 and 297 respectively.

THEOREM 1. *Every two points of a continuous curve are the extremities of at least one simple continuous arc that lies entirely on that curve.*

Proof. Suppose A and B are two points belonging to the continuous plane curve C. Hahn has shown‡ that the curve C is connected "im kleinen," i. e., that if P is a point of C, ϵ is a positive number and K is a circle, of radius $1/\epsilon$, with center at P, then there exists, within K and with center at P, another circle $K_{\epsilon P}$ such that if X is a point within $K_{\epsilon P}$, and belonging to C, then X and P lie together in some connected subset of C that lies entirely within K. Let $\overline{K}_{\epsilon P}$ denote the set of all points $[Y]$ belonging to C such that Y and P lie together in some connected subset of C that lies entirely within K. Clearly $\overline{K}_{\epsilon P}$ contains $K_{\epsilon P}$,

* A set of points is said to be *connected* if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one.

† If S is a point set, S' denotes the set of points composed of S together with all its limit points.

‡ Hans Hahn, "Ueber die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 23 (1914), pp. 318–322.

and indeed if Z is any point of $\overline{K}_{\epsilon P}$ then there exists about Z a circle such that every point of C within this circle belongs to $\overline{K}_{\epsilon P}$. Let $\epsilon = 1$ and for each point P of C construct the corresponding $\overline{K}_{1P}$. Let $\overline{G}_1$ denote the set of all such $\overline{K}_{1P}$'s. I will proceed to show that there exists a simple chain* from A to B every link of which is a point set of the set $\overline{G}_1$. Suppose that no such simple chain exists. Then the points of C fall into two classes, S_A and S_B where S_A is the set of all points $[P]$, belonging to C, such that P can be joined to A by a simple chain every link of which belongs to the set $\overline{G}_1$, while S_B is composed of all the remaining points of C. Since C is connected, one of the sets S_A and S_B contains a point X which is a limit point of the other one. There exists a circle k with center at X such that every point of C that lies within k lies also in $\overline{K}_{1X}$. But the interior of k contains a point A_1 belonging to S_A and a point B_1 belonging to S_B where A_1 is X or B_1 is X according as X belongs to S_A or to S_B. The point A can be joined to A_1 by a simple chain $R_1R_2R_3 \cdots R_n$ every link of which is a point set of the set $\overline{G}_1$. Let R_k be the first link of this chain that has a point in common with $\overline{K}_{1X}$. Then $R_1R_2R_3 \cdots R_k\overline{K}_{1P}$ is a simple chain from A to B_1 every link of which belongs to $\overline{G}_1$. Thus the supposition that A can not be joined to B by a simple chain every link of which is a point set belonging to $\overline{G}_1$ leads to a contradiction. It follows that A can be joined to B by at least one such chain $R_{11}R_{12} \cdots R_{1m_1}$. Call this chain C_1. For each i $(1 \leqq i < m_1)$ select a point P_{1i} common to R_{1i} and R_{1i+1} Let $P_{10} = A$ and $P_{1m_1} = B$. If $0 \leqq i < m_1$ then P_{1i} can be joined to P_{1i+1} by a simple chain C_{1i+1} each link of which is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq 2$ and lies with all its limit points entirely in R_{1i+1}. If any link of the chain C_{11}, except the last one, has a point in common with any link of C_{12}, then omit from C_{11} every link that follows T_{11}, where T_{11} is the first link of C_{11} that has a point in common with a link of C_{12}; also omit

* A *simple chain* from A to B is a finite sequence of point sets $R_1, R_2, R_3, \cdots R_n$ such that (1) R_i contains A if and only if $i = 1$, (2) R_i contains B if and only if $i = n$, (3) if $1 \leqq i \leqq n$, $1 \leqq j \leqq n$, $i < j$, then R_i has a point in common with R_j if and only if $j = i + 1$. The point set R_k $(1 \leqq k \leqq n)$ is said to be the kth link of the chain $R_1R_2R_3 \cdots R_n$ and the chain $R_1R_2R_3 \cdots R_n$ is said to join A to B. Cf. my paper "On the foundations of plane analysis situs," loc. cit., page 134.

from C_{12} every link (if there be any such) that precedes the last link that has a point in common with T_{11}. These omissions having been made, the remaining links of the chains C_{11} and C_{12} form a simple chain $\overline{C}_{12}$ from A to P_{12}. In a similar manner it may be shown that there exists a simple chain $\overline{C}_{13}$ from A to P_{13} such that each link of $\overline{C}_{13}$ is a link of either $\overline{C}_{12}$ or C_{13}. This process may be continued. It follows that there exists a simple chain C_2 from A to B such that each link of C_2 is a link of some C_{1i} $(1 \leqq i \leqq m_1)$. The chain C_2 has the property that each one of its links lies wholly in some single link of the chain C_1 and if a link x of C_2 lies in a link y of C_1 then every link that follows x in C_2 lies either in y or in some link that follows y in C_1. Similarly there exists a chain C_3 having a relation to C_2 analogous to the above indicated relation of C_2 to C_1 and such that every link of C_3 is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq 3$. This process may be continued. Thus there exists an infinite sequence of simple chains $C_1, C_2, C_3, \cdots$ such that (1) each link of the chain C_{n+1} lies, together with all its limit points, wholly in some single link of C_n; (2) if a link x of C_{n+1} lies in a link y of C_n then each link that follows x in C_{n+1} lies either in y or in some link that follows y in C_n; (3) every link of C_n is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq n$.

Let S_n denote the point set which is the sum of all the links of the chain C_n. Let S denote the set of all the points that the sets $S_1, S_2, S_3, \cdots$ have in common. It will be shown that S satisfies Lennes' definition* of a simple continuous arc from A to B.

I. That S is closed and connected follows easily with the help of the lemma on page 233. That it is bounded is evident.

II. To prove that S contains no connected proper subset that contains both A and B, let us first order the points of S. If X_1 and X_2 are two distinct points of S, then there exists n such that if $r \geqq n$ and P is a point of S then X_1 and X_2 do not both lie in $\overline{K}_{rP}$. But every link of C_n is a $\overline{K}_{rP}$ for some point P of C and some $r \geqq n$. Hence for every two distinct points X_1 and X_2 be-

* A *simple continuous arc* from A to B is a bounded, closed, connected set of points containing A and B but containing no connected proper subset that contains both A and B. See N. J. Lennes, "Curves in non-metrical analysis situs with an application in the calculus of variations," *American Journal of Mathematics*, vol. 33 (1911), p. 308, and this BULLETIN, vol. 12 (1906), p. 284.

longing to C there exists n such that X_1 and X_2 do not belong to the same link of C_n. Furthermore it is clear that if X_1 and X_2 do not lie in the same link of C_n but X_1 lies in a link of C_n that precedes one in which X_2 lies, then, if $m > n$, every link of C_m that contains X_1 precedes every link of C_m that contains X_2. The point X_1 is said to *precede* the point X_2 $(X_1 < X_2)$ if there exists n such that every link of C_n that contains X_1 precedes every link of C_n that contains X_2. From facts observed above it follows that if X_1 and X_2 are distinct points of C then either $X_1 < X_2$ or $X_2 < X_1$; while if $X_1 < X_2$ then it is not true that $X_2 < X_1$. Furthermore if $X_1 < X_2$ and $X_2 < X_3$ then $X_1 < X_3$. For there exist n_1 and n_2 such that n_1 and n_2 do not lie in the same link of C_{n_1} and X_2 and X_3 do not lie in the same link of C_{n_2}. Hence every link of $C_{n_1+n_2}$ that contains X_1 precedes every link of $C_{n_1+n_2}$ that contains X_2 and every link of $C_{n_1+n_2}$ that contains X_2 precedes every one that contains X_3. Hence every one that contains X_1 precedes every one that contains X_3. Therefore $X_1 < X_3$.

Suppose now that H is a proper subset of S that contains both A and B. Then there exists a point P belonging to S, but different from A and from B, such that H is a subset of $S - P$. Now $S - P = S_A + S_B$ where S_A is the set of all points of S that precede P and S_B is the set of all points of S that follow P. It is clear that S_A contains A and S_B contains B. Suppose that P_A is a point of S_A. Then there exists n such that every link of C_n that contains P_A precedes every one that contains P. Suppose that some link y of the chain C_n contains P_A and also a point P_B of the set S_B. Since y precedes every link of C_n that contains P, it follows that P_B precedes P, which is contrary to hypothesis. Hence no link of C_n that contains P_A contains any point of S . But some link l of C_n does contain P_A. There exists, about P_A, a circle t such that every point of C that lies within t is a point of l. It follows that there is no point of S_B within the circle t. Therefore P_A is not a limit point of S_B. Similarly no point of S_B is a limit point of S_A. But H contains a point A that belongs to S_A and a point B that belongs to S_B. Moreover H is a subset of $S_A + S_B$. It follows that H is not connected.

It follows that S is a simple continuous arc from A to B. But clearly S is a subset of C. The truth of Theorem 1 is therefore established.

THE UNIVERSITY OF PENNSYLVANIA.

A NOTE ON DISCONTINUOUS SOLUTIONS IN THE CALCULUS OF VARIATIONS.

BY DR. PAUL R. RIDER.

(Read before the Southwestern Section of the American Mathematical Society, December 2, 1916.)

IT is the object of this note to give the corner conditions and the forms of the Carathéodory Ω-function for Bliss's form of the simplest problem of the calculus of variations, and for an analogous form of the problem in space.

For the purpose of orientation and of introducing notation, a brief résumé of a part of the theory of discontinuous solutions as treated by Bolza* is given here.

1. If at a point $P_0(t_0)$ of a curve $x = x(t)$, $y = y(t)$ that minimizes or maximizes the definite integral

$$\int_{t_1}^{t_2} F[x(t), y(t), x'(t), y'(t)]dt$$

the curve possesses a corner, the corner conditions

$$F_{x'}\big|^{t_0-0} = F_{x'}\big|^{t_0+0}, \quad F_{y'}\big|^{t_0-0} = F_{y'}\big|^{t_0+0}$$

must be satisfied.

Let $P_1P_0P_2$ be an extremal (that is, a minimizing or maximizing curve) having a corner at P_0, the corner conditions being satisfied. Suppose that the continuity and other conditions usually imposed in the calculus of variations hold for the arcs P_1P_0, P_0P_2 and for the family of extremals

$$x = \phi(t, a), \quad y = \psi(t, a),$$

which contain the extremal arc $E_0 \equiv P_1P_0$ for $a = a_0$. Designate by τ_0 and $\bar{\tau}_0$ respectively the angles that the positive tangents to the arcs P_1P_0 and P_0P_2 at the point P_0 make with the positive x-axis. Then, if it is desired to find on E_a, a neighboring extremal to E_0, a point $P(t)$ and a direction $\bar{\tau}$ such that τ and τ (τ is the positive direction of E_a at P) satisfy the corner conditions, it is necessary to solve for t and $\bar{\tau}$ the equations

$$(1) \qquad F_{x'} - \bar{F}_{x'} = 0, \quad F_{y'} - \bar{F}_{y'} = 0,$$

* Bolza, Vorlesungen über Variationsrechnung, Chapter 8.

in which the arguments of $F_{x'}$, $F_{y'}$ are $\phi(t, a)$, $\psi(t, a)$, $\cos\tau(t, a)$, $\sin\tau(t, a)$, and of $\bar{F}_{x'}$, $\bar{F}_{y'}$ are $\phi(t, a)$, $\psi(t, a)$, $\cos\bar{\tau}$, $\sin\tau$. The functional determinant of (1) with respect to t, $\bar{\tau}$ at the point P_0 has the value

$$\sqrt{x'^2 + y'^2}\,\bar{F}_1(t_0)\Omega_0,$$

where $\Omega = \bar{F}_x \cos\tau + \bar{F}_y \sin\tau - F_x \cos\bar{\tau} - F_y \sin\bar{\tau}$, and where the $\bar{F}_1$-function* has the arguments x_0, y_0, $\cos\bar{\tau}$, $\sin\bar{\tau}$. The subscript on the Ω-function denotes that the arguments ϕ, ψ, τ, $\bar{\tau}$, wherever they occur in it, have the values x_0, y_0, τ_0, $\bar{\tau}_0$ respectively. Since a usual assumption is that $F_1 \neq 0$ along an extremal, it follows that if $\Omega_0 \neq 0$ equations (1) can be solved uniquely for t and $\bar{\tau}$ and the solution $t = t(a)$, $\bar{\tau} = \bar{\tau}(a)$ will be continuous in the vicinity of $a = a_0$ and will satisfy the initial conditions $t(a_0) = t_0$, $\bar{\tau}(a_0) = \bar{\tau}_0$.

Thus can be obtained a broken extremal $E_a + \bar{E}_a \equiv P_1PP_2$ with a corner at the point P. If a is allowed to vary, a family of such extremals is obtained, and the point P describes a so-called corner curve whose parameter is a.

2. The extension of this theory to the case in which the integrand of the definite integral is a function of three variables and their derivatives is perfectly obvious and is readily accomplished, but it may be worth while to give it here.

For this case there are four equations

$$F_{x'} - \bar{F}_{x'} = 0, \quad F_{y'} - \bar{F}_{y'} = 0, \quad F_{z'} - \bar{F}_{z'} = 0,$$
$$\cos^2\bar{\alpha} + \cos^2\bar{\beta} + \cos^2\bar{\gamma} - 1 = 0, \tag{2}$$

the arguments of $F_{x'}$, $F_{y'}$, $F_{z'}$ being $\phi(t, u, v)$, $\psi(t, u, v)$, $\chi(t, u, v)$, $\cos\alpha(t, u, v)$, $\cos\beta(t, u, v)$, $\cos\gamma(t, u, v)$, and those of $\bar{F}_{x'}$, $\bar{F}_{y'}$, $\bar{F}_{z'}$ being $\phi(t, u, v)$, $\psi(t, u, v)$, $\chi(t, u, v)$, $\cos\bar{\alpha}$, $\cos\bar{\beta}$, $\cos\bar{\gamma}$. At a point $P_0(t_0)$ the value of the functional determinant of the equations with respect to t, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ is found to be

$$2\sqrt{x'^2 + y'^2 + z'^2}\,\sin\alpha_0 \sin\beta_0 \sin\gamma_0\,\bar{F}_1(t_0)\Omega_0,$$

where

$$\Omega = \bar{F}_x \cos\alpha + \bar{F}_y \cos\beta + \bar{F}_z \cos\gamma - F_x \cos\bar{\alpha} - F_y \cos\bar{\beta} - F_z \cos\bar{\gamma}.$$

* For definition see Bolza, loc. cit., p. 196.

The $\bar{F}_1$-function* has the arguments x_0, y_0, z_0, $\cos \bar{\alpha}_0$, $\cos \bar{\beta}_0$, $\cos \bar{\gamma}_0$. If the usual assumption that it is different from zero along an extremal is made, it follows that if $\sin \bar{\alpha}_0 \sin \bar{\beta}_0 \sin \bar{\gamma}_0 \neq 0$, that is, if the extremal arc P_0P_2 does not have an initial direction which is parallel to one of the three axes, and if $\Omega_0 \neq 0$, then the system (2) can be solved uniquely for t, $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ in terms of u, v. A family of broken extremals can be determined as before, and the point P varies on a corner surface whose parameters are u, v.

3. For Bliss's form of the problem† the integral has the form

$$\int_{t_1}^{t_2} f(x, y, \tau) \sqrt{x'^2 + y'^2}\, dt,$$

where $\tau = \arctan y'/x'$. By the methods ordinarily employed the corner conditions that must be satisfied at a point P are found to be

$$f \cos \tau - f_\tau \sin \tau - \bar{f} \cos \bar{\tau} + \bar{f}_\tau \sin \bar{\tau} = 0,$$

$$f \sin \tau + f_\tau \cos \tau - \bar{f} \sin \bar{\tau} - \bar{f}_\tau \cos \bar{\tau} = 0.$$

In these equations the parameter t, of the arguments of f, f_τ, $\bar{f}$, $\bar{f}_\tau$, has been replaced by s, the length of arc.

Their functional determinant with respect to s, $\bar{\tau}$ is

$$\bar{f}_1 \omega,$$

where

$$f_1 = f + f_{\tau\tau}, \quad \omega = \bar{f}_x \cos \tau + \bar{f}_y \sin \tau - f_x \cos \bar{\tau} - f_y \sin \bar{\tau}.$$

Since here also $f_1 \neq 0$, the properties of ω are identical with those of the corresponding function in the Weierstrass form of the problem.

4. In a paper entitled "The space problem of the calculus of variations in terms of angle," to be published in the *American Journal of Mathematics*, I have considered an integral of the form

$$\int_{t_1}^{t_2} f(x, y, \tau, \sigma) \sqrt{x'^2 + y'^2 + z'^2}\, dt,$$

* For definition see Bliss and Mason, "The properties of curves in space which minimize a definite integral," *Transactions Amer. Math. Society*, vol. 9 (1908), p. 442.

† See Bliss, "A new form of the simplest problem of the calculus of variations," *Transactions Amer. Math. Society*, vol. 8 (1907), pp. 405–414.

τ and σ being defined by the relations

$$\frac{x'}{\sqrt{x'^2+y'^2+z'^2}} = \cos\sigma\cos\tau,$$

$$\frac{y'}{\sqrt{x'^2+y'^2+z'^2}} = \cos\sigma\sin\tau,$$

$$\frac{z'}{\sqrt{x'^2+y'^2+z'^2}} = \sin\sigma.$$

Geometrically σ is defined as the angle that the positive tangent to a curve makes with its projection in the xy-plane, and τ is the angle that this projection makes with the positive x-axis. The corner conditions that must be satisfied by the extremals of such an integral are

$$f\cos\sigma\cos\tau - f_\tau\frac{\sin\tau}{\cos\sigma} - f_\sigma\sin\sigma\cos\tau$$
$$-\bar{f}\cos\bar{\sigma}\cos\bar{\tau} + \bar{f}_\tau\frac{\sin\bar{\tau}}{\cos\bar{\sigma}} + \bar{f}_\sigma\sin\bar{\sigma}\cos\bar{\tau} = 0,$$

$$f\cos\sigma\sin\tau + f_\tau\frac{\cos\tau}{\cos\sigma} - f_\sigma\sin\sigma\sin\tau$$
$$-\bar{f}\cos\bar{\sigma}\sin\bar{\tau} - \bar{f}_\tau\frac{\cos\bar{\tau}}{\cos\bar{\sigma}} + \bar{f}_\sigma\sin\bar{\sigma}\sin\bar{\tau} = 0,$$

$$f\sin\sigma + f_\sigma\cos\sigma - \bar{f}\sin\bar{\sigma} - \bar{f}_\sigma\cos\bar{\sigma} = 0.$$

Here again t has been replaced by s.

A somewhat lengthy piece of reckoning gives as the functional determinant of this system, with respect to s, $\bar{\tau}$, $\bar{\sigma}$,

$$-\frac{\bar{f}_1\omega}{\cos\bar{\sigma}},$$

where

$$f_1 = (f\cos^2\sigma - f_\sigma\sin\sigma\cos\sigma + f_{\tau\tau})(f + f_{\sigma\sigma})$$
$$-(f_\tau\tan\sigma + f_{\tau\sigma})^2,$$

$$\omega = \bar{f}_x\cos\sigma\cos\tau + \bar{f}_y\cos\sigma\sin\tau + \bar{f}_z\sin\sigma$$
$$-f_x\cos\bar{\sigma}\cos\bar{\tau} - f_y\cos\bar{\sigma}\sin\bar{\tau} - f_z\sin\bar{\sigma}.$$

If we make the usual assumption $f_1 \neq 0$ (we assume also that $\cos\bar{\sigma} \neq 0$), the ω-function has the same properties as the corresponding function in three-dimensional space.

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THEORY OF THE TOP.

Advisory Committee for Aeronautics. Reports and Memoranda, No. 146. Report on Gyroscopic Theory. By SIR G. GREENHILL. Darling and Son, and T. Fisher Unwin, London, 1914. Folio. iii+277 pp.+49 figs. Boards. Price 10s.

The British Advisory Committee for Aeronautics has printed, or rather, until the war, had printed annual reports which are the prime sources for information regarding the development of the theory and practice of aeroplane design. Unfortunately they no longer divulge their findings. The Committee has also published other reports bearing less directly upon aeronautical problems, but yet advantageous in the more recondite parts of the theory. Of these supplementary publications, Greenhill's report on gyroscopic theory deserves particular attention of the student of mathematics and mechanics, and cannot forever be neglected by the practical engineer, so important are many instances of gyroscopic action becoming.

We are indeed fortunate that the Committee got Greenhill to prepare the report. His long-continued investigations on the top have made him a world-recognized authority on the subject. The fact that we have available for study the great work of Klein and Sommerfeld on the same subject does not in the slightest detract from the value of the present work. The merest glance will convince any reader that the two treatises are conceived in very different ways and that they rather supplement than overlap one another. Besides Greenhill's is decidedly shorter and more concerned with apparatus. It will be of more interest to the student of mechanics and engineering, though of less to the student of the theory of functions of a complex variable. And here we might remark that a beautiful theory and a practical analysis susceptible to ready calculation are unfortunately not always to be combined. The theta function is a thing of beauty, but I have always found the solution of the problem of the motion of the top by theta functions with complex arguments anything but satisfactory from the point of view of calculation. Perhaps after all Legendre knew what he was about when he introduced his third elliptic integral,—at any rate we judge that Greenhill thinks so.

There is a good deal of Greenhillian elliptic analysis in the report, and this may discourage a great many persons; but I desire particularly to emphasize the fact that there is also a great deal of the text which is practically free from elliptic integrals and functions, and which is almost arithmetic and algebraic in its simplicity and directness. The first chapter is on steady gyroscopic motion. Here we have 27 pages that anybody can read. This discussion abounds in the geometric and mechanical illustration for which the author is known. The distinction between strong and weak tops is brought out. The effect of the moving rotating parts of the Gnome engine upon the steering of an aeroplane is shown, also the similar phenomenon in the automobile. Other problems considered are: precession of the equinox, Serson-Fleurias gyroscopic horizon, how to roll a hoop, gyroscopic effect in wheeled carriages and in paddle or screw steamers, stability of an elongated body in a medium (airship or projectile or submarine boat). There is an appendix on general dynamical theory. Naturally, if one is to make this work simple he must concentrate his attention on the vector angular momentum; there is no other way, and the use of this vector is as easy as it is important. Greenhill's first chapter would be valuable for its emphasis on the angular momentum vector even if it contained nothing else of value.

The second chapter is on gyroscopic applications. The simple style of the first chapter is continued. We find treated: the Schlick sea gyroscope, the monorail carriage, the Bessemer saloon, with some mention of steering torpedoes. There is also a section on the theory of oscillation, free, damped, and forced, and an appendix on dynamical units. In both these chapters, as throughout the book, numerous references to the literature are given—to all sorts of literature, not merely to the mathematical.

Chapter III is on the general unsteady motion of the top. The integration rapidly leads to elliptic analysis. The author gives a summary of many ways of treating elliptic integrals and functions. By the mathematician this chapter may be added to the existing tables of formulas. There are, however, other points of interest: the discussion of nearly steady motion, Kirchoff's kinetic analogue of the twisted elastica, the whirling chain, and Clifford's vector treatment of the top. The fourth chapter is largely analytical geometry—Darboux's deformable

hyperboloid, Poinsot motion, curvature and inflexion of the herpolhode, curvature of the polhode, and the like. Chapters V and VI are respectively given to the algebraic cases of top motion, and to numerical illustrations and diagrams. This is the ellipsotomic problem. The cases discussed are: bisection of a period, trisection, quinquesection, seven, nine, and eleven section. Various improvements on the author's work in the *Annals of Mathematics* are given. The analysis will interest the mathematician, the figures will illustrate to anybody a number of cases of motion of the top. The applications are: nutation of the earth's axis, free Eulerian precession and variation of latitude, lunar nutation, and gyroscopic motion of the moon's node. In Chapter VII under the caption, the spherical pendulum, the author treats steady motion and small nutation, and the plane revolving catenary.

With the eighth chapter we come to motion referred to moving origin and axes, and the problems of the motion of a pendulum body and gyroscope about an axle on a whirling arm, of the sphere containing a flywheel and rolling on a table, of the modification of Poinsot motion when the body contains a flywheel spinning on an axle fixed in the body, of the expanding or contracting body, of Kelvin's liquid gyrostat, and finally of the motion of a solid, perforated or not, in an infinite liquid. A knowledge of elliptic functions is not particularly important for the reader of this chapter. In the following ninth and closing chapter Greenhill takes up dynamical problems of steady motion and small oscillations: general motion of a body rolling on a table with attention to steady motion and small oscillations, motion of a body sliding on a smooth table, the single track gyroscopic car, the gyrocompass, Gilbert's barogyroscope, the gyrostatic chain, Kelvin's flexible wire suspension, the gyroscopic pendulum with wire suspension, the stability of a revolving shaft, and the motion of a body with altazimuth suspension. In many of these problems elliptic functions are not mentioned.

It will be seen that the text divides itself naturally into two major parts. Chapters I, II, VIII, and IX deal with practical physical and engineering problems in a practical and on the whole in an elementary way without the use of elliptic functions. There are some sections of the other chapters, III to VII inclusive, which are also elementary, but as a whole these chapters are valuable chiefly to a reader who is not afraid of

detailed elliptic analysis. This twofold nature of the work is desirable, especially when the division is as clean cut as here, because each type of reader can find and use the part of the book he most needs and at the same time he can gain a glimpse, if not an appreciation, of the sort of work on the same problem which interests the other type of reader.

The forty-nine carefully executed figures are collocated upon large plates at the end of the book. For some of the illustrations this arrangement is satisfactory, but others of them should have been placed at the appropriate places in the text for the readers' ease in following demonstrations. The typography of the work seems excellent. We have now in English a great reference book on the top, a worthy companion and counterpart to that of Klein and Sommerfeld in German, one decidedly more compact and cheaper in price.

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NOTES.

THE second annual meeting of the Mathematical Association of America was held at Columbia University, Thursday to Saturday, December 28–30, 1916. Thursday afternoon was devoted to a joint session with the American Mathematical Society, the American Astronomical Society, and Section A of the American Association for the Advancement of Science, and Thursday evening to a joint dinner of the four organizations. The programme of the Association included addresses by FLORIAN CAJORI: "Discussion of fluxions from Berkeley to Woodhouse"; M. W. HASKELL: "University courses in mathematics intended for teachers of secondary mathematics"; T. H. GRONWALL: "A nucleus for a mathematical library"; E. B. WILSON: "The mathematics of aerodynamics." On Friday there was an exhibition of portraits and medals of mathematicians from the collection of David Eugene Smith, the presentation and discussion of the report of the library committee, and a business meeting including the election of the following officers: President: FLORIAN CAJORI; vice-presidents: D. N. LEHMER and OSWALD VEBLEN; secretary-

treasurer: W. D. CAIRNS; members of the council: E. R. HEDRICK, E. V. HUNTINGTON, HELEN A. MERRILL, R. E. MORITZ, D. E. SMITH.

AT the meeting of the Edinburgh mathematical society on December 8 the following papers were read: By L. R. FORD: "On a class of continued fractions"; by E. T. WHITTAKER: "On certain determinants of Cayley and Sylvester"; by E. M. HORSBURGH: "An addition to the slide rule."

THE opening (January) number of volume 39 of the *American Journal of Mathematics* contains the following papers: "Linear difference and differential equations," by TOMLINSON FORT; "Some singularities of a contact transformation," by W. V. LOVITT; "Oscillations near an isosceles-triangle solution of the problem of three bodies as the finite masses become unequal," by DANIEL BUCHANAN; "The finite groups of birational transformations of a net of cubics," by L. C. COX; "On the determination of a certain class of surfaces," by A. E. YOUNG; "On orthoptic and isoptic loci," by HAROLD HILTON and R. E. COLOMB; "A new canonical form for systems of partial differential equations," by L. B. ROBINSON.

THE December number (series 2, volume 18, number 2) of the *Annals of Mathematics* contains the following papers: "The numerical factors of the arithmetic forms $\prod_{i=1}^{n}(1 \pm \alpha_i^m)$," by T. A. PIERCE; "A problem in geometry connected with the analytic continuation of a power series," by T. H. GRONWALL; "On the power series for $\log(1+z)$," by T. H. GRONWALL; "On the convergence of Binet's factorial series for $\log \Gamma(s)$ and $\Psi(s)$," by T. H. GRONWALL; "Note on Cauchy's integral formula," by J. L. WALSH; "On the rational, integral invariants of nilpotent algebras," by O. C. HAZLETT; "On trigonometric series," by W. L. HART.

JOHN WILEY and Sons announce the publication of volumes on Elliptic Integrals by HARRIS HANCOCK and Differential and Integral Calculus by H. B. PHILLIPS.

PROFESSORS PAUL PAINLEVÉ, of the University of Paris, and VITO VOLTERRA, of the University of Rome, have been elected honorary members of the Royal Institution, of London.

PROFESSOR TOMLINSON FORT, of the University of Michigan, has been appointed head professor of mathematics in the University of Alabama.

MR. E. J. OGLESBY, of the University of Virginia, has been appointed professor of mathematics in William and Mary College.

DR. P. H. LINEHAN has been promoted to an assistant professorship of mathematics at the College of the City of New York.

MR. J. K. WHITTEMORE has been appointed instructor in mathematics in the Sheffield Scientific School.

MR. J. A. BULLARD and Mr. C. E. NORWOOD have been appointed instructors in mathematics in the U. S. Naval Academy.

CHARLES SMITH, master of Sidney Sussex College and author of well-known textbooks on algebra and analytic geometry, died November 13, at the age of seventy-two years.

PROFESSOR H. A. SAYRE, of the University of Alabama, died December 2. Professor Sayre had been a member of the American Mathematical Society since 1891.

CHARLES A. PITKIN, professor of mathematics and physics at Thayer Academy, South Braintree, Mass., since its opening in 1877, died November 5, aged sixty-three years.

BOOK CATALOGUES:—Henry Sotheran and Company, 140 Strand, London, catalogue 766, exact sciences, about 1800 titles.—John Wheldon and Company, 38 Great Queen Street, London, scientific transactions and periodicals, 800 titles.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BOLZANO (B.). Werke herausgegeben von A. Höfler. I. Wissenschaftslehre in 4 Bänden. Band 1: Versuch einer ausführlichen und grössten Theils neuen Darstellung der Logik mit steter Rücksicht auf deren bisherige Bearbeiter. Mit einer Vorrede des J. C. A. Heinroth. Erster Band, Sulzbach, 1837. Leipzig, F. Meiner, 1914. 8vo. 16+571 pp. Geb. M. 14.00

CAVALCANTI (A.). Mathématique ou logique. Essai d'un cours philosophique de calcul arithmétique d'après Auguste Comte. Traduction française par Auguste de Aranjo Gonçalves. Paris, Blanchard, 1916. 8vo. 25+827 pp. Fr. 14.00

COMTE (A.). See CAVALCANTI (A.).

DARBOUX (G.). See POINCARÉ (H.).

GONÇALVES (A. DE A.). See CAVALCANTI (A.).

HALSTED (G. B.). See LOBATSCHEVSKI (N.).

HEINROTH (J. C. A.). SEE BOLZANO (B.).

HÖFLER (A.). See BOLZANO (B.).

HOLBA (S.). Eine neue Bahn in das Reich der Algebra. Forschungen und Gedanken eines Unberufenen. Algebraische Lösung des Casus irreducibilis, der Gleichung 5. Grades und des Fermat'schen Problems. Budapest, Kommissionsverlag Ludwig Kókai, 1915. 64 pp. 8 vo. M. 2.50

KNOBLAUCH (J.). See WEIERSTRASS (K.).

KÖRNER (K.). Die Mathematik im Preussischen Lehrerbildungswesen. (Abhandlungen über den mathematischen Unterricht in Deutschland veranlasst durch die internationale mathematische Unterrichtskommission, Band 5, Heft 7.) Leipzig, Teubner, 1916. 136 pp. M. 1.60

LEBON (E.). See POINCARÉ (H.).

LEWY (H.). Ueber die apriorischen Elemente der Erkenntnis. I: Die Stufen der reinen Anschauung. Erkenntnistheoretische Untersuchungen über den Raum und die geometrischen Gestalten. Leipzig, F. Meiner, 1914. 9+204 pp. 8vo. M. 6.00

LIETZMANN (W.). Die Ausbildung der Mathematiklehrer an den höheren Schulen Deutschlands. (Berichte und Mitteilungen veranlasst durch die internationale mathematische Unterrichtskommission. Erste Folge, XI.) Leipzig, Teubner, 1915. 14 pp.

LOBATSCHEVSKI (N.). Geometrical researches on the theory of parallels. Translated from the original by G. B. Halsted. New edition. Chicago, Open Court, 1914. $1.25

MACAULAY (F. S.). The algebraic theory of modular systems. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 19.) Cambridge, University Press, 1916. 4s. 6d.

MACMAHON (P. A.). Combinatory analysis. Volume 2. Cambridge, University Press, 1916. Royal 8vo. 340 pp. 18s.

NÖRLUND (N. E.). See POINCARÉ (H.).

POINCARÉ (H.). Oeuvres publiées par G. Darboux. Tome 2 publié avec la collaboration de N. E. Nörlund et de E. Lebon. Paris, Gauthier-Villars, 1916. 4to. 77+628 pp. Fr. 35.00

RIEBESELL (P.). Die mathematischen Grundlagen der Variations- und Vererbungslehre. (Mathematische Bibliothek, Nr. 24.) Leipzig, Teubner, 1916. 8vo. 45 pp. M. 0.80

ROTHE (R.). See WEIERSTRASS (K.).

WEIERSTRASS (K.). Mathematische Werke herausgegeben unter Mitwirkung einer von den königlichen Preussischen Akademie der Wissenschaften eingesetzten Kommission. Band 5: Vorlesungen über die Theorie der elliptischen Functionen. Bearbeitet von J. Knoblauch. Band 6: Vorlesungen über Anwendungen der elliptischen Functionen. Bearbeitet von R. Rothe. Mit einem Bildnis von Weierstrass. Berlin, Mayer und Müller, 1915. 8+327+9+354 pp. 4to.

II. ELEMENTARY MATHEMATICS.

BAUER (W.) und HANXLEDEN (E. v.). Oberstufe der Arithmetik. Braunschweig, Vieweg, 1913. 8vo. 8+208 pp. Geb. M. 3.60

——. Unterstufe der Arithmetik. Braunschweig, Vieweg, 1913. 8vo. 10+271 pp. Geb. M. 4.00

BRENKE (W. C.). See LONG (E.).

BÜTZBERGER (F.). Lehrbuch der ebenen Trigonometrie mit vielen Aufgaben und Anwendungen. Zürich, O. Füssli, 1916. 8vo. 98 pp. Fr. 2.00

——. Lehrbuch der Stereometrie. Zürich, O. Füssli, 1916. 8vo. 119 pp.

CRATHORNE (A. R.). See RIETZ (H. L.).

HANXLEDEN (E. v.). See BAUER (W.).

HEDSTRÖM (J. S.) och RENDAL (C.). Algebra för gymnasiet. Stockholm, A. Bonnier, 1915. 8vo. 245 pp.

JACKSON (L. L.). See YOUNG (J. W. A.).

LONG (E.) and BRENKE (W. C.). Plane geometry. (Correlated mathematics for secondary schools, Part 2.) New York, Century Company, 1916. 12mo. 8+276 pp. Cloth. $1.00

RENDAL (C.). See HEDSTRÖM (J. S.).

RIETZ (H. L.), CRATHORNE (A. R.) and TAYLOR (E. H.). School algebra. (American Mathematical Series.) Complete in one volume. New York, Holt, 1916. 8vo. 15+506 pp. $1.25

TAYLOR (E. H.). See RIETZ (H. L.).

TRIPARD (L.). Les mathématiques après l'école primaire. Paris, Dunod et Pinat, 1916. 12mo. 376 pp. Cartonné. Fr. 5.00

YOUNG (J. W. A.) and JACKSON (L. L.). A plane geometry. (Twentieth Century Text.) New York, Appleton, 1913.

III. APPLIED MATHEMATICS.

BEGGS (G. E.). Live load stresses in railway bridges. With formulas and tables. New York, Wiley, 1915. 5+123 pp. $2.00

BENOIT (C.) See LANCHESTER (F. W.).

BROWN (J. C.). See FINNEY (H. A.).

BRYAN (C. W.). See JOHNSON (J. B.).

CABRAL (A. F.). Cuadro sinoptico de la doctrina biocosmica de la gravitacion universal y de la jeneracion de los mundos. Santo Domingo, Imp. La Cuna de América, 1915. Folio. 17 pp.

CARY (E. R.). Geodetic surveying. New York, Wiley, 1916. 9+272 pp. Cloth. $2.50

CONKLIN (C. D.). Structural steel drafting and elementary design. New York, Wiley, 1915. 7+154 pp. Cloth. $2.50

FAWDRY (R. C.). Dynamics. Part 1. London, Bell, 1916. 8+177+9 pp. 3s.

FINNEY (H. A.) and BROWN (J. C.). Modern business arithmetic. Complete course. New York, Holt, 1916. 8vo. 6+488 pp. $1.10

GALÉ (J. G.). Matemáticas financieras. Primera parte: Intereses y anualidades ciertas. Buenos Aires, Libreria de Antonio Garcia Santos. Moreno y Bolivar, 1916. 226 pp.

GILTAY (J. W.). De strijkinstrumenten. Leyden, A. W. Lijthoff, 1916. 11+103 pp.

JOHNSON (J. B.), BRYAN (C. W.) and TURNEAURE (F. E.). The theory and practice of modern framed structures. Part 3: Design. 9th edition, rewritten by F. E. Turneaure and W. S. Kinne. New York, Wiley, 1916. 12+486 pp. Cloth. $4.00

KINNE (W. S.). See JOHNSON (J. B.).

KRÜGER (L.). Transformation der Koordinaten bei der konformen Doppelprojektion des Erdellipsoids auf die Kugel und die Ebene. Leipzig, Teubner, 1914. 43 pp. 4to. M. 3.00

LANCHESTER (F. W.). Le vol aérien. Aérodonétique. Traduit de l'anglais sur la deuxième édition par C. Benoit. Paris, Gauthier-Villars, 1916. 8vo. 17+478 pp. Fr. 14.00

MELAN (J.) and STEINMAN (D. B.). Plane and reinforced concrete arches. New York, Wiley, 1915. 10+161 pp. Cloth. $2.00

MERRIMAN (M.). American civil engineers' pocket book. Edited by M. Merriman and sixteen associate editors. 3d edition, revised and enlarged. New York, Wiley, 1916. 1600 pp. Leather. $5.00

MOULTON (F. R.). An introduction to astronomy. Revised edition. New York, Macmillan, 1916. Cr. 8vo. 568 pp. $2.25

PHILIP (G.). Planisphere showing the principal stars visible for every hour in the year. London, G. Philip and Son, 1916. 1s. 6d.

ROHRBERG (A.). Theorie und Praxis des Rechenschiebers. (Mathematische Bibliothek, Nr. 23.) Leipzig, Teubner, 1916. 8vo. 50 pp. M. 0.80

ROWARTH (E.). Elements of engineering drawing. London, Methuen, 1916. 12+131 pp. 2s. 6d.

STEINMAN (D. B.). See MELAN (J.).

TURNEAURE (F. E.). See JOHNSON (J. B.).

WADDELL (J. A. C.). Bridge engineering. 2 volumes. New York, Wiley, 1916. 68+2177 pp. $10.00

WOLFF (G.). Mathematik und Malerei. (Mathematische Bibliothek, Nr. 20–21.) 1 volume. Leipzig, Teubner, 1916. 8vo. 76 pp.+4 plates. M. 1.60